Reducible correlations in all SLOCC equivalent W states

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Plan...

- **Quantum Marginal Problem (QMP)**
  - QMP
  - Weaker version(s) of QMP
- **Reducible Correlations**
  - Optimal RC
  - Relevance
- **Bipartite RC in SLOCC equivalent W states**
  - All SLOCC equivalent W states
  - Are determined by ρ^{1K}
  - Are determined by $\rho^{K(K+1)}$
- **Discussion**
  - Issues and some other approches
  - Conclusion

Quantum Marginal Problem (QMP)

Given two density matrices ρ^A and ρ^B for two systems A and B, there always exists a density matrix $\rho^{AB} = \rho^A \otimes \rho^B$ for the composite system AB *compatible* with these reduced density matrices (RDM).

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No, not always!

Examples:

- $\nexists\ |\psi\rangle_{AB}$ if $\lambda(\rho^A) \neq \lambda(\rho^B)$ (Schmidt decomposition).
- $\nexists$ any ρ^{ABC} compatible with $\rho^{AB} = \rho^{BC} = |\psi^-\rangle\langle\psi^-|$.
- $\nexists$ any $|\psi\rangle_{ABCD}$ having all bipartite RDMs I_4.

An N-partite System

$$\rho$$

An N-partite System

ρ

An N-partite System

ρ

$\Downarrow$

It has $\binom{N}{N-1} = N$ number of $(N-1)$-partite RDMs

An N-partite System

$$\rho$$

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It has $\binom{N}{K}$ number of K-partite RDMs

An N-partite System

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It has $\binom{N}{2} = \frac{N(N-1)}{2}$ number of bipartite-RDMs

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✪ Easier to fall $\Downarrow$ ☺, but very difficult to rise $\Uparrow$ ☹.

QMP: The quantum jigsaw puzzle

Given a set of RDMs to find a compatible ρ

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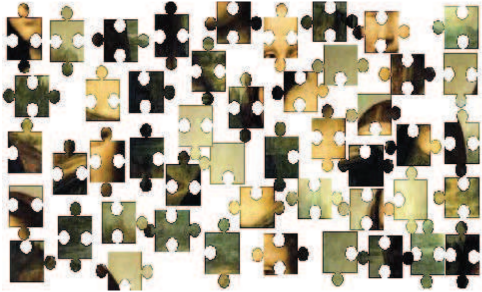
For a given set of Parts...



QMP: The quantum jigsaw puzzle

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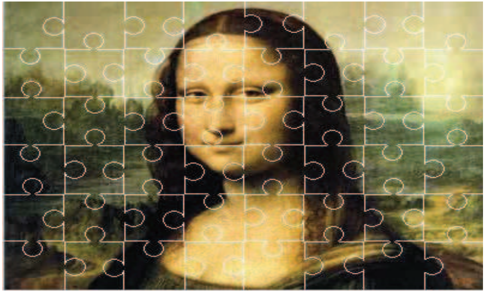
Not always..., but sometimes they can be seen to be the parts of a



QMP: The quantum jigsaw puzzle

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QMP in a Nutshell

- Pure QMP asks for the existence of a compatible pure state $\rho = |\psi\rangle\langle\psi|$, while mixed QMP asks for constraints on the spectra.
- The compatibility conditions should be in terms of spectral inequalities.
- Some problems of the same spirit:
  - Many-body Physics: Local Hamiltonian Problem
  - Quantum Chemistry: N-representability Problem
- Higuchi, Sudbery and Szulc (2003) have solved the 1-RDM N-qubit pure QMP, Bravyi (2004) solved the 2-qubit mixed QMP, Coleman ($\approx$ 1963) solved the 1-RDM mixed N-representability and Alexander Klyachko (2005) solved the 1-RDM pure N-representability.

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QMP is QMA-complete![1]

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- The original state can be taken as pure state as we can always purify a mixed state.
- This apparently simple question has many non-trivial consequences in Quantum Information Theory (some will be mentioned).

(Ir)reducible Correlations

- Since ρ^A and ρ^B are from $|\psi\rangle_{AB} = \sum \lambda_i |ii\rangle$, $\exists$ another compatible $|\phi\rangle_{AB} = \sum \pm\lambda_i |ii\rangle$. Thus the RDMs ρ^A and ρ^B can not *determine* any entangled $|\psi\rangle_{AB}$ uniquely, or generic bipartite pure states are undetermined by their RDMs.

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- So it is reasonable to think that $|\psi_{ABC\ldots}\rangle$ can be determined by its RDMs iff it can not be written as $\sum \lambda_i |ii\ldots\rangle$, i.e. iff it is not LU to $|GGHZ\rangle = \sum \lambda_i |ii\ldots\rangle$!!

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- Yes, it indeed is correct!

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A (usually pure) state ρ, is said to be

- K-reducible, if it can be determined by K-RDMs
- K-irreducible, if K-reducible but can not be determined by $(K-1)$-RDMs
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- Classical probabilities are irreducible: p_{ijk} and $q_{ijk} = p_{ijk} + (-1)^{\epsilon_{ijk}}\delta$ share the same marginals.
- $|GGHZ\rangle$ and LU are irreducible, all other 3-qubit pure states are 2-reducible (Linden et al, PRL 2002).
- This holds for N-qubits too (Walck and Lyons, PRL 2008).
- and even for arbitrary N-partite states, restricting within pure states (Feng et al, QIC 2009).

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- *Almost all* N-qudit pure states are $(\lceil N/2 \rceil + 1)$-reducible (Jones and Linden, PRA 2005)
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- *Generalized* N-qubit Dicke states $|GD_N^\ell\rangle = \sum a_k \mathrm{Perm}|11\ldots 1_\ell 00\ldots 0\rangle$ are 2ℓ-reducible (PP and S. Rana, JPA 2009) and actually optimal $(\ell + 1)$-reducible (PP and S. Rana, PRA 2011).
- *Standard* N-qubit Dicke states $|D_N^\ell\rangle = \sum \mathrm{Perm}|11\ldots 1_\ell 00\ldots 0\rangle$ are 2-irreducible (Chen et al., arXiv:1106.1373v2).

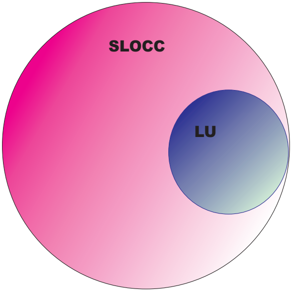
- RC essentially exhibits peculiarity of quantum correlation as compared to its classical analogue.
- Study of RC will lead to the understanding of different types of correlations that a multipartite state can exhibit–which will lead to classification of quantum states. This is a fundamental area of study in quantum information theory (QIT).
- The study will explore the traditional field of many-body quantum physics in terms of correlation among various parties (e.g., characterizing ground states of local Hamiltonian).
- The undetermined states can be exploited in the quantum secret sharing scheme (Hsieh et al., EPJD 2011).

RC/IC under SLOCC

It follows that two LU equivalent states are either irreducible or reducible (according to whether either of them is LU to $|GGHZ\rangle$). Thus we can say that under LU, reducibility remains preserved. From the early definition/basic understanding, entanglement remains preserved under LU but can change drastically under more general operations e.g., SLOCC. So, its natural to ask, how the reducibility changes under SLOCC.

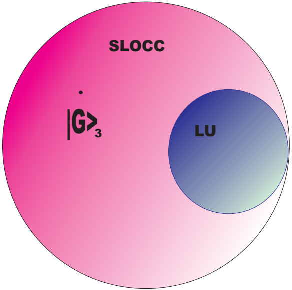
RC/IC under SLOCC

As LU $\subset$ SLOCC, $\exists$ states which are SLOCC but not LU to $|GGHZ\rangle$.



RC/IC under SLOCC

$|G\rangle_3 = 1/\sqrt{2}(|W\rangle + |\widetilde{W}\rangle)$ is an example (Usha Devi et al., arXiv:1002.2820).



The SLOCC operator

$$A = -\frac{1}{\sqrt[6]{3}} \begin{pmatrix} 1 & \omega \\ 1 & \omega^2 \end{pmatrix}$$

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But the second most well known state, namely the W sate is not SLOCC to $|GHZ\rangle$. So, what about the reducibility of all SLOCC equivalent W state?

$|W\rangle$ has optimal 2-reducibility. Do they preserve it?

All SLOCC equivalent W states [Kintas and Turgut, JMP 2010]

- Any N-qubit pure state which is SLOCC equivalent to the standard W state is given by $|\psi\rangle = \otimes_{k=1}^{N} A_k |W_N\rangle$ where A_ks are any invertible operators. If

$$A_k = \begin{bmatrix} \alpha_k & \gamma_k \\ \beta_k & \delta_k \end{bmatrix}$$

then A_k transforms $|0\rangle_k \to \alpha_k|0\rangle_k + \beta_k|1\rangle_k \equiv |u\rangle_k$,
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- $$|\psi\rangle = \frac{1}{\sqrt{N}}(|vu\ldots u\rangle + |uv\ldots u\rangle + \ldots + |uu\ldots v\rangle)$$

- Now A_k is invertible $\Rightarrow$ $\{u_k, v_k\}$ are LI and so can be extended to an orthonormal basis of $\mathcal{H}^k$ i.e., $\exists$ orthonormal $\{p_k, q_k\}$ s.t.

$$\begin{aligned} |p\rangle_k &= a_k|u\rangle_k \\ |q\rangle_k &= b_k|u\rangle_k + b'_k|v\rangle_k \end{aligned}$$

- Thus $|\psi\rangle$ can be written as

$$|\psi\rangle = z_0|pp\dots p\rangle + \sum_{k=1}^{N} z_k|pp\dots p_{k-1}q_k p_{k+1}\dots p\rangle$$

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- Clearly the bases can be redefined to *absorb* the phases in the complex coefficients and $|\psi\rangle$ can be written as

$$|\psi\rangle = c_0|0\dots 0\rangle + \sum_{k=1}^{N} c_k|0\dots 0_{k-1}1_k 0_{k+1}\dots 0\rangle, \quad (1)$$

$c_k \geq 0, \quad \sum_{k=0}^{N} c_k^2 = 1$ (for normalization).

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- Though we can consider all $c_k \geq 0$, we will not restrict–we will consider all c_k as complex numbers satisfying normalization. Also, for our purpose, WLOG at least 3 $c_k \neq 0$. Thus (1) is the general form of all SLOCC equivalent states $|GW\rangle$.

All SLOCC equivalent W states are determined by $\{\rho^{1K}\}$

Theorem1

All SLOCC equivalent W states are uniquely determined by only $(N-1)$ number of bipartite RDMs $\{\rho^{1K}, \quad K = 2, 3, \ldots, N\}$.

Proof

1. From (1), we readily have

$$\rho^{1K}_{GW} = \begin{bmatrix} n_{1K} & c_0\overline{c_K} & c_0\overline{c_1} & 0 \\ & |c_K|^2 & c_K\overline{c_1} & 0 \\ & & |c_1|^2 & 0 \\ & & & 0 \end{bmatrix}$$

   where $n_{1K} = 1 - |c_1|^2 - |c_K|^2$ by normalization.

2. Now, if possible, let another N-qubit density matrix (possibly mixed, thereby subscript M)

$$\rho_M = \sum_{i_1,\ldots,j_N=0}^{1} r_{(i_1\ldots i_N)(j_1\ldots j_N)} |i_1\ldots i_N\rangle\langle j_1\ldots j_N| \equiv \sum r_{IJ}|I\rangle\langle J|$$

share the same bipartite RDMs with $|GW\rangle$ i.e., $\rho_M^{1K} = \rho_{GW}^{1K} \quad \forall K = 2(1)N$.
To be a valid physical state, the entries of ρ_M must satisfy

- i). $\overline{r_{IJ}} = r_{JI}$ (for hermiticity)
- ii). $\sum_I r_{II} = 1$ (for normalization $\mathrm{Tr}(\rho_M) = 1$)
- iiia). $r_{II} \geq 0 \quad \forall I$ and $r_{II} = 0 \Rightarrow r_{IJ} = 0 \forall J$.
- iiib). All principle minors of ρ^M are ≥ 0
- iiic). Particularly, $|r_{IJ}|^2 \leq r_{II} r_{JJ}$

3a. Since there exists no term $|11\rangle\langle 11|$ in ρ_{GW}^{1K}, we must have $r_{(1 i_2 i_3\ldots 1_k \ldots i_N)(1 i_2 i_3 \ldots 1_k \ldots i_N)} = 0$ and hence by property (iiia) of PSD matrices, we have

$$r_{(1 i_2 \ldots 1_k \ldots i_N)(j_1 j_2 \ldots j_N)} = r_{(i_1 i_2 \ldots i_N)(1 j_2 \ldots 1_k \ldots j_N)} = 0$$

for all $i_1, i_2, \ldots, i_N, j_1, j_2, \ldots, j_N = 0, 1$.

3b. Comparing the coefficient of $|10\rangle\langle 10|$ from ρ_M^{1N} and ρ_{GW}^{1N}, it follows that

$$r_{(10\ldots0)(10\ldots0)} = |c_1|^2$$

4a. Now consider the non-diagonal element $|01\rangle\langle 10|$ of ρ_M^{1K} and ρ_{GW}^{1K}, $\quad K = 2(1)N$. It follows that

$$r_{(0\ldots0_{K-1}1_K0_{K+1}\ldots0)(100\ldots0)} = c_K\overline{c_1}$$

and hence by the property (iiic) of PSD matrices with $I = (0\ldots0_{K-1}1_K0_{K+1}\ldots0)$ and $J = (100\ldots0)$ we have

$$r_{(0\ldots0_{K-1}1_K0_{K+1}\ldots0)(0\ldots0_{K-1}1_K0_{K+1}\ldots0)} \geq |c_K|^2$$

4b. Similarly, comparing the coefficients of $|00\rangle\langle 10|$, it follows that

$$r_{(00\ldots0)(00\ldots0)} \geq |c_0|^2$$

4c. From normalization, the property (iiia) of PSD matrices it follows that all the above inequalities will be equalities; and each r_{II} in which I has two or more 1, is zero. So, by property (iiia), $r_{IJ} = r_{JI} = 0$ whenever I or J has two or more 1.

4d. Comparing the coefficients of $|00\rangle\langle 01|$ from ρ_M^{1K} and ρ_{GW}^{1K}, we have

$$r_{(00...0)(00...01_K 0...0)} = c_0 \overline{c_K}, \quad \forall K = 2(1)N$$

4e. Thus, collecting all the results it follows that ρ_M has the same form as $|GW\rangle\langle GW|$ and they share the same diagonal elements, same elements along the row and column $(00\ldots0)$ and $(10\ldots0)$. The only remaining task is to prove

$$r_{(0...01_J 0...0)(0...01_K 0...0)} = c_J \overline{c_K} \text{ for } J > K = 1(1)(N-1)$$

This part is quite difficult, because no further condition can arise from sharing of the RDMs.

5. If $c_J\overline{c_K}=0$ then by property (iiia), our requirement follows trivially. Hence let us assume $c_J\overline{c_K}\neq 0$. To complete the proof we will now apply property (iv) to ρ_M. Let us consider the following principle minor consisting of the rows and columns $(0\ldots01_J0\ldots0),(0\ldots01_K0\ldots0),(10\ldots0)$:

$$\begin{vmatrix} |c_J|^2 & r & c_J\overline{c_1} \\ \overline{r} & |c_K|^2 & c_K\overline{c_1} \\ \overline{c_J}c_1 & \overline{c_K}c_1 & |c_1|^2 \end{vmatrix}$$

where $r=r_{(0...01_J0...0)(0...01_K0...0)}$. The value of this determinant is[a]

$$-|c_J|^2|c_K|^2|c_1|^2|1-\frac{r}{c_J\overline{c_K}}|^2$$

Since this should be non-negative, we have $r=c_J\overline{c_K}$. ■

[a] To evaluate easily, divide first row by c_J, first column by $\overline{c_J}$, second row by c_K, second column by $\overline{c_K}$, third row by c_1 and third column by $\overline{c_1}$.

All SLOCC equivalent W states are determined by $\{\rho^{K(K+1)}\}$

Theorem2

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Outline of the proof

- First we note that no basis of ρ_M can have two consecutive 1, means

$$r_{(i_1 i_2 \ldots i_{K-1} 1_K 1_{K+1} i_{K+2} \ldots i_N)(j_1 j_2 \ldots j_N)} = 0$$

Next we will show that no basis of ρ_M can have the sequence 101 i.e.

$$r_{(i_1 \ldots 1_K 0_{K+1} 1_{K+2} \ldots i_N)(j_1 j_2 \ldots j_N)} = 0$$

For simplicity, let us first take $K = 1$ and the other cases will follow similarly. So, comparing the diagonal elements $|01\rangle\langle 01|$, $|10\rangle\langle 10|$ and the off-diagonal elements $|01\rangle\langle 10|$ from ρ^{12}_M and ρ^{12}_{GW} we have (keeping in mind that no basis can have two consecutive 1)

$$\begin{aligned}
\sum_{i_4,i_5,\dots,i_N=0}^{1} r_{(010i_4i_5\dots i_N)(010i_4i_5\dots i_N)} &= |c_2|^2 \\
\sum_{i_3,i_4,\dots,i_N=0}^{1} r_{(10i_3i_4\dots i_N)(10i_3i_4\dots i_N)} &= |c_1|^2 \\
\sum_{i_4,i_5,\dots,i_N=0}^{1} r_{(010i_4i_5\dots i_N)(100i_4i_5\dots i_N)} &= c_2\overline{c_1}
\end{aligned}$$

- Considering absolute values in the last equation, we have

$$\sum_{i_4,i_5,\dots,i_N=0}^{1} |r_{(010i_4i_5\dots i_N)(100i_4i_5\dots i_N)}| \geq |c_2||c_1|$$

- Again, by the PSD property (iiic),

$$\sum_{i_4,i_5,\dots,i_N=0}^{1} |r_{(010i_4\dots i_N)(100i_4\dots i_N)}| \leq |c_2||c_1|$$

- Equality $\Rightarrow$

$$\sum_{i_4,\ldots,i_N=0}^{1} r_{(100 i_4 \ldots i_N)(100 i_4 \ldots i_N)} = |c_1|^2$$

$$r_{(101 i_4 i_5 \ldots i_N)(101 i_4 i_5 \ldots i_N)} = 0$$

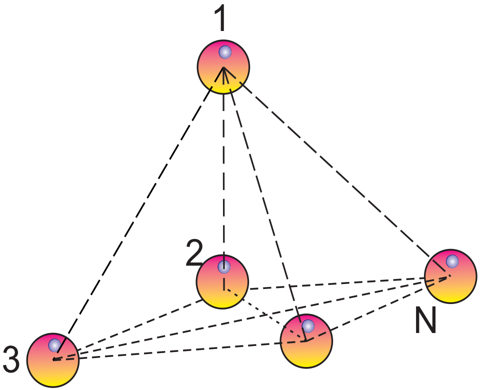
Similarly all $\rho_M^{K(K+1)}$ and $\rho_{GW}^{K(K+1)}$: $\nexists$ sequence $101 \Rightarrow$

$$\sum_{i_5,i_6,\ldots,i_N=0}^{1} r_{(0100 i_5 \ldots i_N)(1000 i_5 \ldots i_N)} = c_2 \overline{c_1}$$

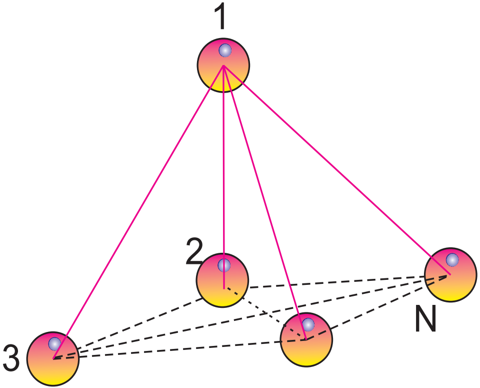
$$r_{(1001 i_5 \ldots i_N)(j_1 j_2 \ldots j_N)} = 0$$

- Next ordered RDMs (ρ^{23} and ρ^{34}) gives $i_5 = 0 = i_6$. Thus $\nexists$ sequence 1001, 10001, so on....i.e. only possible sequence are those having only one 1 and no one. $\blacksquare$

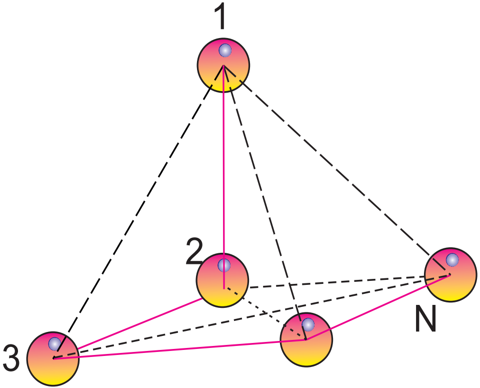
Each pair of qubits in a generic $|W\rangle$ states is correlated



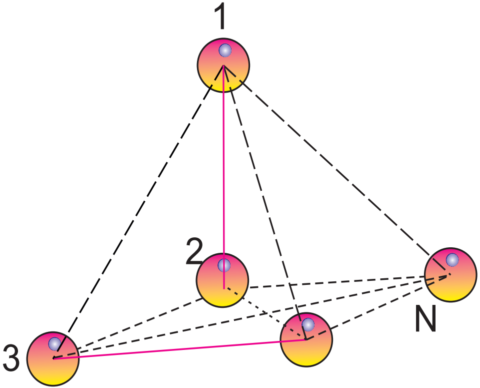
But they can be determined uniquely by the bipartite RDMs $\rho^{12}, \rho^{13}, \ldots, \rho^{1N}$.



Or by the bipartite RDMs $\rho^{12}, \rho^{23}, \ldots, \rho^{(N-1)N}$......but



can not be determined by the bipartite RDMs $\rho^{12}, \rho^{34}, \ldots, \rho^{(N-1)N}$.



Optimal reducibility of G states $|G\rangle = \frac{1}{\sqrt{2}}(|W\rangle + |\widetilde{W}\rangle)$

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Theorem3

For $N \geq 6$, the N-qubit generic G state

$$|GG_N\rangle = \sum_{K=1}^{N} (a_K|01_K0\ldots0\rangle + b_K|10_K1\ldots1\rangle)$$

(with $\sum(|a_K|^2 + |b_K|^2) = 1$, $a_K b_K \neq 0$) is uniquely determined, among arbitrary states, by only **two** $(N-2)$-partite RDMs, but can not be determined by lower order RDMs.

Optimal reducibility of *generalized* Dicke states $|GD_N^\ell\rangle$

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$|GD_N^\ell\rangle$ is uniquely determined, among arbitrary states, by its $(\ell+1)$-partite marginals $\rho_{GD}^{1P_2P_3\ldots P_{\ell+1}}, \quad P_k \in \{2,3,4,\ldots,N\}$.

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Note

If we consider the *standard* Dicke states (i.e., $|GD_N^{\ell}\rangle$ with all equal coefficients), then only bipartite marginals are sufficient to determine them.

Conclusion

- Any pure state which is SLOCC equivalent to W states can be determined (among arbitrary states) from its bipartite marginals. Moreover any *correlated* set of $(N-1)$ number of bipartite marginals (e.g. $\{\rho^{1K}\}, \{\rho^{K(K+1)}\}$) suffices. This reveals that all information in such states is imprinted into these few RDMs. This feature of W states is absent in GHZ-type states.
- Generalized Dicke states $|GD_N^\ell\rangle$ are optimal $(\ell+1)$-reducible. The standard Dicke states are optimal 2-reducible.
- For higher qubits (more than five), the G states are optimal $(N-2)$-reducible.

Discussion

Issues

- Can a mixed state have reducible correlations? If so, what are those states?
- Determine all optimal K-reducible states.
- Or, at least determining optimal reducibility of well known states (e.g., Werner and cluster states) also of interest.

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- Majorana Representation [A. R. Usha Devi et al., arXiv:1003.2450v1]: They have shown that N qubit states containing two distinct spinors are $(N-1)$-reducible. May give some insight into the general problem.

Thank You